Causal Models

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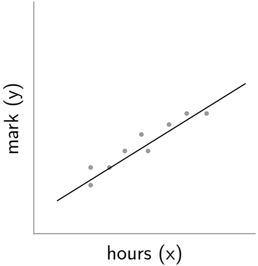
Example to have a mind

- consider writing down a mathematical model for causal situations
- here's an example to consider: firing squad
  - there is a court, which may order the execution of a prisoner
  - if the court orders the captain signals
  - if the the captain signals two separate rifleman will fire killing the prisoner
- how could we evaluate statements like:
  - "if the prisoner is dead, then even if one rifeman witheld, the prisoner would be dead"
  - the key word of counterfactuals: "would"

A second example to keep in mind: why regression models are not causal

- a second example: hours studied and grades
  - students study a number of hours for an exam, x
  - students take an exam and receives a mark, y
- regression does not distinguish between $y \approx f(x)$ or $x \approx g(y)$ (but you could estimate g)



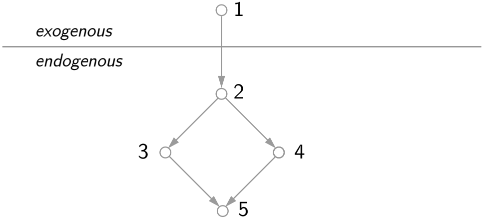
Basic Concepts

Typed Graph

- let (V, E) a graph and let $\{A_v\}_{v \in V}$ a collection of sets
  - call $(V, E, \{A_v\})$ a *typed graph*
  - for vertex $v \in V$, call A_v the *domain* of v
  - for subset of vertices $U \subset V$, denote product of domains (w.r.t. fixed order) by $A_U = \prod_{u \in U} A_u$
- if (V, E) is directed
  - call $\{u \in V \mid (u, v) \in E\}$ the *parents* of v; denote the parents of v by P_v
  - call $v \in V$ *exogenous* if P_v is empty, otherwise call v *endogenous*

Example: (Directed) Typed Graph

- example with $V = \{1, 2, 3, 4, 5\}$ and $E = \{(1, 2), (2, 3), (2, 4), (3, 5), (4, 5)\}$



- and boolean domains $A_v = \{0, 1\}$ for each $v = 1, 2, 3, 4, 5$

Graphical Variable Model

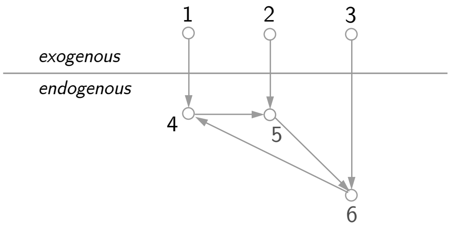
- let $(V, E, \{A_v\})$ a typed directed graph
- let X a subset of the domain of exogenous vertices
- let $f_v : A_{P_v} \to A_v$ for each endogenous vertex
- call $(V, E, \{A_v\}, X, \{f_v\})$ a *graphical variable model*
  - call elements of V *variables*
  - call elements of X *circumstances*
  - call f_v the *dynamics* for variable v

Interpretation: Graphical Variable Model

- let $(V, E, \{A_v\}, X, \{f_v\})$ a graphical variable model
- interpretation: a system of equations defined by relations in $\{f_v\}$ and structure in E
  - denote the product domain of the endogenous variables by Y
  - let $F : X \times Y \to Y$ such that $F_v(x, y) = F_v(z) = f_v(z_{P_v})$
- for fixed x, call solutions y of $F(x, y) = y$ the *outcomes*
  - may be none, one or many solutions
  - corresponds to root finding of $G(y; x) = F(x, y) - y$
  - leads to question: when will we know there will be one unique outcome?

Example: Graphical Variable Model

- example with $V = \{1,2,3,4,5,6\}$ and $E = \{(1,4),(2,5),(3,6),(4,5),(5,6),(6,4)\}$



  - real domains $A_v = \mathbf{R}$ for each $v = 1,2,3,4,5,6$
  - dynamics $f_4(x_1, y_3) = x_1 + a_{13}y_3$, $f_5(x_2, y_1) = x_2 + a_{21}y_1$, and $f_6(x_3, y_2) = x_3 + a_{32}y_2$

- outcomes are solutions, for fixed x, of

$$F(x,y) = x + \begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix} y = y$$

Causal Model

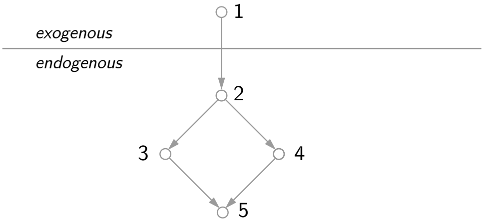
- motivation: if (V, E) of graphical variable model is acyclic, then
  - there exists a unique solution to system of equations for fixed exogenous values
  - computational implication: topologically sort graph, set exogenous variables and evaluate f_v
- definition: call $(V, E, \{A_v\}, X, \{A_v\})$ with (V, E) acyclic a *causal graphical variable model*
  - call it a *causal model* for short
  - call $f : X \to Y$, *outcome map*, denoting the product domain of the exogenous variables by Y
  - call $f(x)$ the *outcome* of circumstance x
  - call graph of f the *possibilities*; denote the graph of f by Γ_f

Interpretation: Causal Model

- given a set of (endogenous) variables to model and (exogenous) variables external to model
- given specified values for these exogenous variables (circumstances)
- use the model to determine the values for endogenous variables (outcomes)
- computationally: topologically sort the graph, then evaluate the f_v

Example: Causal Model

- same typed graph as before, with circumstances $X = \{0, 1\}$ for vertex 1



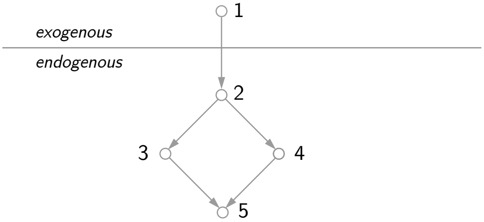
- specify dynamics functions for each of the vertices
  - $f_2, f_3, f_4 \equiv \mathrm{id}$ (the identity function)
  - $f_5(a, b) = a \vee b$ (the logical or function)
- use circumstances and dynamics to find set of possibilities $\{(0, 0, 0, 0, 0), (1, 1, 1, 1, 1)\}$

Evidence, Intervention, and Counterfactual Model

- let $(V, E, \{A_v\}, X, \{f_v\})$ with outcome map $f : X \to Y$
  - let U a set of endogenous vertices and $\{\phi_u : A_{P_u} \to A_u\}_{u \in U}$, call $(U, \{\phi_u\})$ an *intervention*
  - let $W \subset V$ and $w \in A_U$, call (U, w) *evidence*
- define a *counterfactual* causal model $(G, X', \{f'_v\})$ for evidence (E, e) and an intervention $(U, \{\phi_u\})$
  - $X' = \{z_{V_x} \mid z \in \Gamma_f \text{ and } z_E = e\}$;
    - interpretation: only include circumstances consistent with the evidence
  - $f'_v = \phi_v$ if $v \in U$ and $f'_v = f_v$ otherwise
    - interpretation: change dynamics of variables in U, do not change structure E

Example: Counterfactual Model

- causal model as before, with circumstances $X = \{0, 1\}$ for vertex 1,
  - and dynamics $f_2, f_3, f_4 \equiv \text{id}$ and $f_5(a, b) = a \vee b$

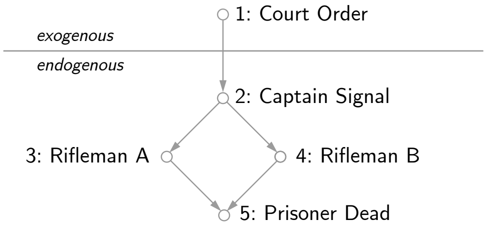


- use evidence $(\{5\}, (0))$ and intervention $(\{3\}, \{\phi_3 \equiv 1\})$
  - only circumstance consistent with evidence is (0)
  - intervention fixes variable 3 at value 1
- using "new" dynamics we find only possibility of counterfactual model is $(0, 0, 1, 0, 1)$

Example: Firing Squad

Example: Firing Squad Interpretation

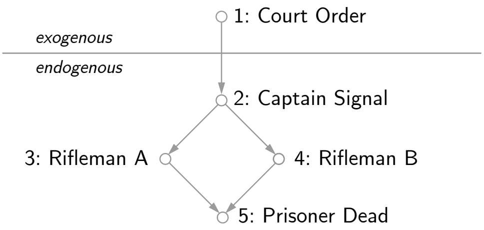
- causal model as before; attach firing squad interpretation



- each boolean variable corresponds to indicator of the action or event
- in english, "if the court orders, the captain signals and the rifleman (A and B) fire, killing the prisoner"
- two possibilities: $\{(0,0,0,0,0),(1,1,1,1,1)\}$
  - circumstance (0): court witholds, captain witholds, riflemen withold, prisoner lives
  - circumstance (1): court orders, captain signals, riflemen shoot, prisoner dies

Example: Prediction

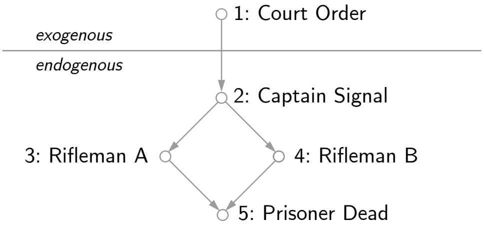
- same causal model, with outcome map f



- for all $z \in \Gamma_f$, $\neg z_3 \implies \neg z_5$
  - in english, "if rifleman A did not shoot, then the prisoner is alive"
  - example of *prediction*, as in all orders of V, $3 \prec 5$

Example: Abduction

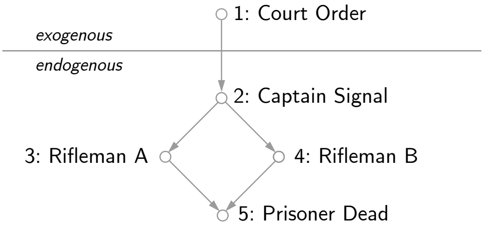
- same causal model, with outcome map f



- for all $z \in \Gamma_f$, $\neg z_5 \implies \neg z_2$
  - in english, "if the prisoner is alive, then the captain did not signal"
  - example of *abduction*, as in all orders of V, $5 \succ 2$

Example: Transduction

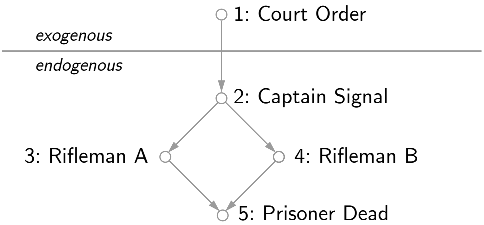
- same causal model, with outcome map f



- for all $z \in \Gamma_f$, $z_3 \implies z_4$
  - in english, "if rifleman A shot, then rifleman B shot"
  - example of *transduction*, as there exists an order in which $3 \prec 4$ and one in which $3 \succ' 4$

Example: Intervention

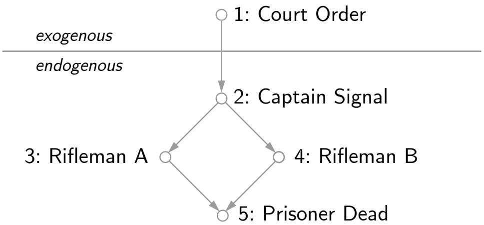
- *intervention* causal model, with outcome map g; intervention $(\{3\}, \{\phi_3 \equiv 1\})$,



- for all $z \in \Gamma_g$, $\neg z_2 \implies \neg z_4 \wedge z_5$
  - in english, "if the captain witholds, but rifleman A shoots, then rifleman B witholds and the prisoner dies"
  - example of an *action* modifying the model, as normally rifleman A follows the captain

Example: Counterfactual

- *counterfactual* causal model with outcome map g; evidence $(\{5\}, (1))$, intervention $(\{3\}, \{\phi_3 \equiv 0\})$



- for all $z \in \Gamma_g$, z_5
  - in english, "if the prisoner is dead, then even if rifleman A withheld, the prisoner would be dead"
  - example of a *counterfactual*, as rifleman A did not in fact withold

Parameters & Probabilities

Parameterized Graphical Variable Model

- let Θ a set
- let $\{(V, E, \{A_v\}, X, \{f_v(\cdot;\theta)\})\}_{\theta\in\Theta}$ a familiy of graphical variable models
- call $(V, E, \{A_v\}, X, \{f_v(\cdot;\theta)\})$ a *parameterized graphical variable model*
  - call θ the *parameters*

Probabilistic Graphical Variable Model

- let $(V, E, \{A_v\})$ a typed graph
- let X a subset of the product domain of exogenous variables and $\mathcal{X}$ a σ-alebgra on X
- let Y the product domain of endogenous variables and $\mathcal{Y}$ a σ-algebra on Y
- let $\mathbf{P}_X : \mathcal{X} \to [0,1]$ a probability measure on $(X, \mathcal{X})$
- let $f_v : A_{P_v} \to A_v$ measurable for v endogenous
- call $(V, E, \{A_v\}, \mathbf{P}_X, \{f_v\})$ a *probabilistic graphical variable model*
  - call $\mathbf{P}_X$ the *exogenous distribution*
  - denote the measure $\mathbf{P}_X \circ f$ by $\mathbf{P}_Y$; call it the *endogenous distribution*
  - let $(\Gamma_f, \mathcal{X} \times \mathcal{Y})$ the product measurable space with induced measure $\mathbf{P}$; call $\mathbf{P}$ the *model distribution*
- interpretation: identify each vertex with a random variable

Parameterized Probabilistic Causal Model

- let Θ a set
- let $\{(V, E, \{A_v\}, \mathbf{P}_X(\cdot; \theta), \{f_v(\cdot; \theta)\})\}_{\theta \in \Theta}$ a family of probabilistic causal models
- call $(V, E, \{A_v\}, \mathbf{P}_X(\cdot; \theta), \{f_v(\cdot; \theta)\})$ *parameterized probabilistic causal model*
  - call θ the *parameters*
  - denote the model distribution by $\mathbf{P}_\theta$, indicating the dependence on the parameters

Notions akin to Causation

Influence

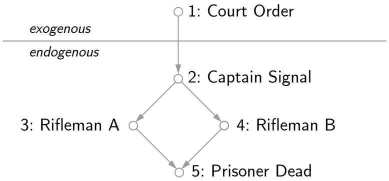
- let $f : X \to Y$ the outcome map of a causal model and x a circumstance
  - let $v \in V$, and $U \subset V$ with $v \notin U$
  - let $(\{U\}, \{\phi_u\})$ an intervention inducing outcome map g
  - denote $(x, f(x))$ by z and $(x, g(x))$ by $\tilde{z}$
- if there exists an intervention on U such that $z_v \neq \tilde{z}_v$, we say U *influences* v in circumstance x
  - additionally, we say U influences v if it influences v in at least one circumstance
- **proposition:** if $\{u\}$ influences v, then u is an ancestor of v
  - a simple necessary condition on structure for influence
  - **corollary:** if variable v is exogenous then it has no influencers
- **proposition:** if U influences v and $U \subset W$, then W influences v

Responsibility

- let $f : X \to Y$ the outcome map of a causal model and x a circumstance
  - let $v \in V$ boolean (*i.e.*, $A_v = \{0, 1\}$), $U \subset V$ with $v \notin U$ and $z_v = 1$
- if U influences v in x we say U is *responsible* for s in x
  - influence requires changing the value of z_v, which in this case has only two options
  - interpretation: intervening on U could have *prevented* v in circumstance x
- **proposition:** if U responsible and $U \subset W$, then W is responsible
  - interpretation: any set containing a responsible set is responsible
- if there exists $Q \subset U$, $Q \neq U$ such that Q is responsible, we call U *reducible*
  - if U is not reducible we call it *irreducible*

Influence & Responsibility Example: Firing Squad

- consider $v = z_5$ and circumstance (1); in this circumstance z_5 is 1



- any subset of $\{1, 2, 3, 4\}$ influence 5; in this circumstance, responsibility is more limited
- both $\{1\}$ (court) and $\{2\}$ (captain) are responsible; both irreducible
- neither $\{3\}$ (rifleman A) nor $\{4\}$ (rifleman B) is responsible
  - however, $\{3, 4\}$ (set of rifleman A and B) is responsible for the prisoner's death; final irreducible set
- this example disproves the following "chain-of-responsibility" proposition:
  - if $\{u\}$ is responsible for v, $\exists$ path $((u, v_1), (v_1, v_2), \ldots, (v_p, v))$ with $\{v_i\}$ is responsible for v for $i = 1, \ldots, p$

Multiple Responsible Sets

Problem of Multiple Responsible Sets

- let s a boolean variable in a causal model taking value 1 in circumstance x
- problem: there are generally several responsible $U \subset V$ for s in x
  - simple issues:
    - can have multiple responsible sets of the same cardinality
    - can have multiple different interventions corresponding to each responsible set
  - subtle issues:
    - U and W have same cardinality, but variables in W pre-empt those in U
    - $U \subset W$ and $U \neq W$ but the intervention certifying W's responsibility is "more reasonable"
- does concept of reducibility go far enough?
  - prior example $\{2, 3, 4\}$ has irreducible responsible subsets $\{2\}$ and $\{3, 4\}$

Naive Solution of Multiple Responsible Sets

- let $h : \mathcal{I} \to \mathbf{R}$, where $\mathcal{I}$ denotes the set of interventions, and define the order $\preceq$ on $\mathcal{I}$ by

$$(U, \{\phi_u\}) \preceq (\tilde{U}, \{\phi_{\tilde{u}}\}) \text{ if and only if } h((U, \{\phi_u\})) \leq h((\tilde{U}, \{\phi_{\tilde{u}}\})).$$

- example: cardinality ordering
  - fix $r : A_V \to [0, 1)$ and define $h((U, \{\phi_u\})) = |U| + r(z)$
    - where $z = (x, g(x))$ and g is outcome map corresponding to intervention
  - if $|U| \leq |\tilde{U}|$, then $(U, \{\phi_u\}) \preceq (\tilde{U}, \{\phi_{\tilde{u}}\})$
  - interpretation: order by cardinality first, then by rating r
- example: distance to evidence
  - let (E, e) evidence and (A_E, d) a metric space, define $h((U, \{\phi_u\})) = d(e, z_E)$
- example: likelihood ordering
  - fix P a distribution on A_v and define $h((U, \{\phi_u\})) = -\log(P(z)))$
    - where $z = (x, g(x))$ and g is outcome map corresponding to intervention
  - interpretation: order by likelihood of possibility induced by intervention

Minimal Responsible Set: Problem

- define "minimal" responsible sets U as solutions of

$$
\begin{aligned}
\textbf{minimize} \quad & h((U, \{\phi_u\})) \\
\textbf{subject to} \quad & z_v = 0 \\
& z = (x, g(x)) \\
& g \text{ is outcome map for } (U, \{\phi_u\}) \\
& U \subset V - \{v\} \text{ and } \phi_u : A_{P_u} \to A_u
\end{aligned}
$$

  with decision variable $(U, \{\phi_u\})$

- interpretation: find the "smallest" intervention preventing $z_v = 1$ in circumstance x
- the equality constraint $z_v = 0$ certifies that U is responsible for v in x
- challenging: $O(2^{|V|})$ possible responsible sets, ϕ_u need not live in finite dimensional space

Minimal Responsible Set: Simplification

- **proposition:** w.l.o.g. can consider constant interventions $\phi_u \equiv c_u$ for $c_u \in A_u$
- can write equivalent problem

$$
\begin{array}{rl}
\textbf{minimize} & h((U, \{\phi_u\})) \\
\textbf{subject to} & z_v = 0 \\
& z = (x, g(x)) \\
& g \text{ is outcome map for } (U, \{\phi_u \equiv c_u\}) \\
& U \subset V - \{v\} \text{ and } c_u \in A_u
\end{array}
$$

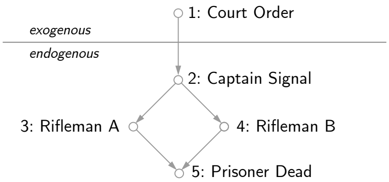
- interpretation of decision variables: choose intervention points U and values $\{c_u\}$

Parents Mediate Responsibility

- **motivation:** want to use a local property about responsibility to make a global statement
- **proposition:** if P_s is not a responsible set for s, then there is no responsible set for s in $V - \{s\}$
  - in fact, a refinement holds: if $\nexists Q \subset P_s$ responsible, then $\nexists R \subset V - \{s\}$
- interpretation: if the parents are not responsible, then no one is responsible
- intuition: all responsibility has to go through the parents
- contrapositive: if there exists $R \subset V - \{s\}$ responsible for s, then $\exists Q \subset P_s$ responsible for s

Firing Squad Example: Parents Mediate Responsibility

- consider $v = z_5$ and circumstance (1); in this circumstance z_5 is 1



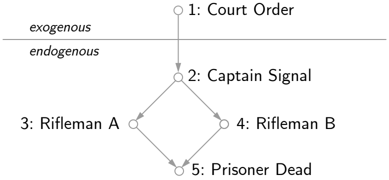
- we saw that $\{1\}$ and $\{2\}$ are responsible
- proposition tells us that $\{3, 4\}$ is responsible
- if (not true here) no intervention on $\{3, 4\}$ would change z_5, no intervention on $\{1\}$ and $\{2\}$ would

Derivative Responsibility

- suppose U is a responsible set for boolean variable s
  - partition U into U_x and U_n
  - denote set $W = \{v \in V \mid v \in P_u \text{ for } u \in U_n\}$
- if $U_x \cup W$ is responsible for s then we call U *derivative*
- if U is not derivative, then we call it *original*
- **proposition:** if $U \subset V_x$ is responsible for s, then U is original
  - interpretation: an exogenous intervention is always original
- existence of responsible set equivalent to responsibility of the parent set
  - originality of parent set allows us to ignore rest of graph

Firing Squad Example: Derivative Responsibility

- consider $v = z_5$ and circumstance (1); in this circumstance z_5 is 1



- we saw that $\{1\}$, $\{2\}$, and $\{3, 4\}$ are responsible
- only the set $\{1\}$ is original (obvious: it only contains exogenous variables)
- the sets $\{2\}$ and $\{3, 4\}$ are derivative
  - $\{3, 4\}$ can be derived from intervening on $\{2\}$
  - $\{2\}$ can be derived from intervening on $\{1\}$

Structural Equation Models

Structural Equation Model

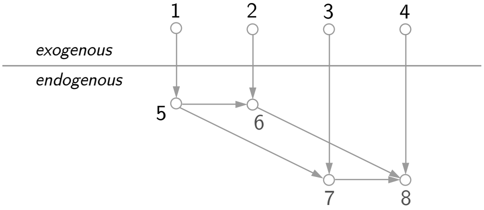
- a *structural equation model (SEM)* is a probabilistic causal model
- it has p mutually independent exogenous variables, each with one child
  - *i.e.*, there is one exogenous variable corresponding to each endogenous variable
- call these p exogenous variables the *noise* variables
- call subgraph (U, F) where U is the set of endogenous vertices and $F := \{(u, v) \in E \mid u, v \in U\}$ *endogenous subgraph*

(Linear) Additive Structural Equation Model (with Gaussian Errors)

- let $(V, E, \{A_v\}, \mathbf{P}_X, \{f_v\})$ a SEM, denote the endogenous parents of v by $\bar{P}_v$
- if $f_v(z_{\bar{P}_v}, \varepsilon_v) = \sum_{u \in \bar{P}_v} f_{(u,v)}(z_u) + \varepsilon_v$, we call it an *additive structural equation model*
  - if the exogenous distribution of an additive SEM is Gaussian we call it an *additive SEM with Gaussian errors*
  - if the dynamics of an additive SEM are linear, *i.e.*, $f_{(u,v)}(a) = \theta_{(u,v)} a$, we all it a *linear additive SEM*
- if both the exogenous distribution is Gaussian and the dynamics linear, we call it a *linear gaussian SEM*

Example: linear Gaussian structural equation model

- let $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $E = \{(1, 2), (2, 6), (3, 7), (4, 8), (5, 6), (5, 7), (6, 8), (7, 8)\}$



- let $\mathbf{P}_X$ be $N(0, 1)$ and dynamics
  - $f_5(\varepsilon_1) = \varepsilon_1$
  - $f_6(\gamma_1, \varepsilon_2) = \theta_1\gamma_1 + \varepsilon_2$
  - $f_7(\gamma_2, \varepsilon_3) = \theta_2\gamma_2 + \varepsilon_3$
  - $f_8(\gamma_2, \gamma_3, \varepsilon_4) = \theta_3\gamma_2 + \theta_4\gamma_3 + \varepsilon_4$

Structure Learning

Structure Learning

- frequently called "causal inference" or "causal discovery"
- let $\{(V, E, \{A_v\}, \mathbf{P}_X(\cdot;\theta), \{f_v(\cdot;\theta)\})\}_{\theta\in\Theta}$ a family of probabilistic causal models
- **question:** how to go from $\mathbf{P}$ to (θ, E) in class of parameterized causal models
- wait: assumes access to $\mathbf{P}$?!
  - often results are given with this assumption
  - to go from data to structure, first go from data to $\mathbf{P}$

Structures & Representation

- let V a set (of vertices)
- call the set $\mathcal{E} = \{E \in V \times V \mid (V, E) \text{ directed, acyclic}\}$ the *structures* on V
- let $\{((V, E), \mathbf{P}_X(\cdot; \theta), \{f_v(\cdot; \theta)\})\}_{\theta \in \Theta}$ a family of probabilistic causal models
  - for $\theta \in \Theta$, $E \in \mathcal{E}$, denote the model distribution of $((V, E), \mathbf{P}_X(\cdot; \theta), \{f_v(\cdot; \theta)\})$ by $\mathbf{P}_\theta$
  - likewise for θ, E', denote the model distribution by $\mathbf{P}'_\theta$
- if $(E, \theta) \in \mathcal{E} \times \Theta$ has model distribution $\mathbf{P}_\theta$ we that (E, θ) *represents* $\mathbf{P}$

Faithfulness

- let $(V, E, \{A_v\}, \mathbf{P}_X(\cdot;\theta), \{f_v(\cdot;\theta)\})$ with model distribution $\mathbf{P}_\theta$
- then the model class is *faithful* if there does not exist E, E', θ such that $\mathbf{P}'_\theta = \mathbf{P}_\theta$
- interpretation: once we have fixed a structure then no edges "disappear" by choice of θ
- examples of faithfulness failing exist even for linear gaussian models

Minimality

- define a relation on $\mathcal{E}$ where $E \preceq E'$ if for all $\theta \in \Theta$ there exists $\theta' \in \Theta$ such that $\mathbf{P}_\theta = \mathbf{P}'_{\theta'}$
- interpretation: E precedes E' if every distribution representable by E is representable by E'
- if $E \in \mathcal{E}$ and there does not exist $E' \neq E$ with $E' \preceq E$ then we call E *minimal*
- interpretation: choose the simplest structure representing E
  - simplest means representing the fewest possible distributions

Minimal Representing Structure

- let $\{((V, E), \mathbf{P}_X(\cdot;\theta), \{f_v(\cdot;\theta)\})\}_{\theta\in\Theta}$ a family of probabilistic causal models
- let $\mathbf{P}$ a distribution on $(\Gamma_f, \mathcal{X} \times \mathcal{Y})$
- call the set $\mathcal{R}(\mathbf{P}) = \{E \in \mathcal{E} \mid \exists\theta \in \Theta$ so that (E, θ) represents $\mathbf{P})\}$ the *representing* structures
- if $E \in \mathcal{R}(\mathbf{P})$ is minimal we call it the *minimal representing strcture*

Bounded Linear Example

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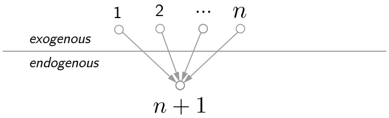
- parameters of model: positive integer n and vector $\alpha \in \mathbf{R}^n$
- typed graph $V = \{1, \ldots, n+1\}$ and $E = \{(1, n+1), (2, n+1), \ldots, (n, n+1)\}$

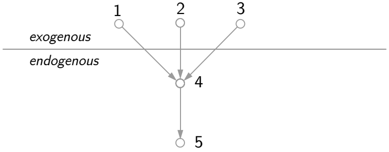


and domains $A_i = [0, 1]$ for $i = 1, \ldots, n$ and $A_{n+1} = [1, n]$

- circumstances $X = [0, 1]^n$
- dynamics $f_{n+1}(x_1, \ldots, x_{n+1}) = \sum_i \alpha_i x_i$
- $\{i\}$ influences $n+1$ if $\alpha_i \neq 0$

Bounded Linear Example: All Responsible

- consider $n = 3$ and $\alpha = 1$ basic bounded linear model with one additional boolean node (5)

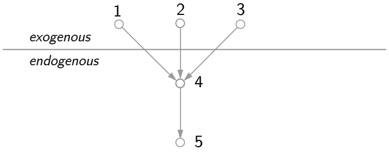


so the dynamics are $f_4(a, b, c) = a + b + c$ (sum) and $f_5(a) = 1$ if $a \geq \tau$ (indicator of threshold)

- consider circumstance $(1, 1, 1)$ and $\tau = 2.5$
  - any subset of $\{1, 2, 3, 4\}$ is responsible for 5
  - intuitively, $\{1\}$ $\{2\}$ and $\{3\}$ are each individually responsible
  - weird artifact of model: $\{4\}$ is responsible
    - could have defined dynamics of 5 directly as $f_5(a, b, c) = 1$ if $a + b + c \geq \tau$

Bounded Linear Example: One Responsible

- consider $n = 3$ and $\alpha = 1$ basic bounded linear model with one additional boolean node (5)

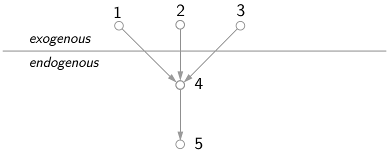


so the dynamics are $f_4(a, b, c) = a + b + c$ (sum) and $f_5(a) = 1$ if $a \geq \tau$ (indicator of threshold)

- consider circumstance $(1.0, 0.1, 0.1)$ and $\tau = 0.5$
  - again, $\{4\}$ is de facto responsible
  - but now only $\{1\}$ is responsible, not $\{2\}$ or $\{3\}$
    - justification for 2 and 3 not responsible: $1 + \xi + .1 \geq 0.5$ for all $\xi \geq 0$

Bounded Linear Example: Two Responsible

- consider $n = 3$ and $\alpha = 1$ basic bounded linear model with one additional boolean node (5)



so the dynamics are $f_4(a, b, c) = a + b + c$ (sum) and $f_5(a) = 1$ if $a \geq \tau$ (indicator of threshold)

- consider circumstance $(1, 1, 1)$ and $\tau = 1.5$
  - again, $\{4\}$ is de facto responsible
  - now, none of $\{1\}$ $\{2\}$ or $\{3\}$ are responsible
    - justification for no individual responsibility: $1 + 1 + \xi \geq 1.5$ for all $\xi \geq 0$
  - however, the sets $\{1, 2\}$, $\{2, 3\}$ and $\{1, 3\}$ are each responsible